The single risk factor approach to capital charges in case of correlated loss given default rates

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Abstract

A new methodology for incorporating LGD correlation effects into the Basel II risk weight functions is introduced. This methodology is based on modelling of LGD and default event with a single loss variable. The resulting formulas for capital charges are numerically compared to the current proposals by the Basel Committee on Banking Supervision.

1 Introduction

Up to now (January 2004), the proposals of the Basel Committee on Banking Supervision (BCBS) for new rules of regulatory capital requirements (Basel II) are implicitly based on an assumption of independent loss given default (LGD) rates and default events. Since the actual risk weight formulas were derived by means of a transition to an asymptotic limit that eliminated diversifiable – and in particular independent – components, LGDs enter the risk weight formulas only by their expected values. However, over the past years evidence of correlation effects between LGDs and default events became stronger (cf. Frye, 2000, 2003).

In the literature (Pykhtin, 2003), an approach to incorporating LGD correlation into the Basel II one-factor model is suggested which models LGDs as *potential losses* that are defined but not observed also in case of the obligor being non-defaulted. Apart from this conceptual difficulty, another problem of the potential losses approach follows from the need to simultaneously estimate an additional – compared to the Basel II model – correlation parameter.

By modelling LGD dependence via the conceptually different *single risk factor* approach, this note suggests a way to avoid the problems which are encountered with the model by Pykhtin. In particular, the single risk factor model does not require additional correlation parameters but can be fed with LGD volatilities whose values can be statistically estimated by the banks or

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prudentially fixed by the supervisory authorities. Additionally, the model shares with Pykhtin's model the property of extending the Basel II model in a way that takes into account the case of defaulted obligors (i.e. obligors whose probability of default equals 1).

This note is organized as follows: The single risk factor model of correlated LGDs is derived in Section 2. In particular, Equation (2.10a) is ready for application as generic form of a capital charge formula that recognizes LGD correlations. The question how to calibrate the model is discussed in Section 3. Numerical illustrations of the concept are provided in Section 4. Section 5 summarizes and concludes the note. In Appendix A, we discuss in more detail the connection of the single risk factor approach with the potential loss model.

2   The single risk factor approach to regulatory LGD modelling

The basic idea for the single risk factor approach is to describe with a single loss variable the loss suffered with an obligor. This is in contrast to the potential loss model (Pykhtin, 2003) where default event and amount of loss are specified with separate random variables.

Consider a fixed obligor and denote by L the loss as percentage of the total exposure which results from granting credit to the obligor. Denote with p the obligor's probability of default. Then L will be zero with probability $1 - p$ and will take positive values with probability p. Formally, we describe the probability distribution function F_L of L as

$$F_L(t) \;=\; \mathrm{P}[L \leq t] \;=\; 1 - p + p\,F_D(t), \tag{2.1}$$

where $p = \mathrm{P}[L > 0]$ is the obligor's probability of default and $F_D(t) = \mathrm{P}[L \leq t \,|\, L > 0] = p^{-1}\,\mathrm{P}[0 < L \leq t]$ is the distribution function of the observed losses. Denote with Φ the standard normal distribution function. Then, in a Basel-II-like framework, L can be represented as

$$L \;=\; F_L^*\big(\Phi(\sqrt{\rho}\,X + \sqrt{1 - \rho}\,\xi)\big) \tag{2.2}$$

with independent, standard normally distributed random variables X and ξ, *asset correlation*[1] $\rho \in [0, 1]$, and the *generalized inverse* or *quantile* function F_L^* of F_L. In general, the quantile function G^* of a distribution function G is defined as

$$G^*(z) \;=\; \min\big\{t : \; G(t) \geq z\big\}. \tag{2.3}$$

If G is continuous and strictly increasing, then G^* coincides with the common inverse function of G, i.e. we have $G^* = G^{-1}$. Applied to F_L^*, definition (2.3) yields the representation

$$F_L^*(z) \;=\; \begin{cases} 0, & \text{if } z \leq 1 - p, \\ F_D^*\big(\frac{z-1+p}{p}\big), & \text{if } z > 1 - p \end{cases} \tag{2.4}$$

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[1] The variable $\sqrt{\rho}\,X + \sqrt{1 - \rho}\,\xi$ is commonly interpreted as minus the change in value of the obligor's assets within a fixed period of time. If the loss expressed by the change is too large, the obligor defaults. In order to express the dependence between different obligors, the change variable is decomposed into a *systematic* and an *idiosyncratic* part. The correlation ρ of two change variables is called *asset correlation*. In technical terms, (2.2) may be interpreted as a *copula*-representation of the model (cf. Nelsen, 1999).

for the quantile function of the obligor's loss distribution where F_D^* again has to be determined according to (2.3). Together, (2.2) and (2.4) imply

$$L = \begin{cases} 0, & \text{if } \sqrt{\rho}\, X + \sqrt{1-\rho}\, \xi \leq \Phi^{-1}(1-p), \\ F_D^*\big(\frac{\Phi(\sqrt{\rho}\, X + \sqrt{1-\rho}\, \xi) - 1 + p}{p}\big), & \text{otherwise.} \end{cases} \tag{2.5}$$

(2.5) can be equivalently written as the product of the indicator function of the default event $\sqrt{\rho}\, X + \sqrt{1-\rho}\, \xi > \Phi^{-1}(1-p)$ and the factor $F_D^*\big(\frac{\Phi(\sqrt{\rho}\, X + \sqrt{1-\rho}\, \xi) - 1 + p}{p}\big)$. The second factor can – similarly to Pykhtin's model – be interpreted as loss in case of default.

From representation (2.1) of the loss distribution follows that in the single risk factor model the expected loss may be calculated as the product of the probability of default and the expected loss given default, i.e.

$$\mathrm{E}[L] = p\, \mathrm{E}[L \,|\, L > 0] = p \int_0^\infty t\, F_D(dt). \tag{2.6a}$$

In order to be able to calculate capital charges in the Basel II sense, we need in addition an evaluable representation of $\mathrm{E}[L \,|\, X = x]$ (see, e.g., Gordy, 2003, for an explanation). From (2.5) follows

$$\mathrm{E}[L \,|\, X = x] = \int_{\frac{\Phi^{-1}(1-p) - \sqrt{\rho}\, x}{\sqrt{1-\rho}}}^{\infty} \varphi(z)\, F_D^*\left(\frac{\Phi(\sqrt{\rho}\, x + \sqrt{1-\rho}\, z) - 1 + p}{p}\right) dz, \tag{2.6b}$$

where $\varphi(z) = \big(\sqrt{2\pi}\big)^{-1} e^{-1/2\, z^2}$ denotes the standard normal density. The case of an already defaulted obligor ($p = 1$) is admitted in (2.6a) and (2.6b). The lower integration limit in (2.6b) then has to be taken as $-\infty$. Note that the choice of F_D as

$$F_D(t) = \begin{cases} 0, & \text{if } t < LGD, \\ 1, & \text{if } t \geq LGD. \end{cases} \tag{2.7a}$$

for some constant LGD will bring us back to the Basel II framework of the risk weight functions (see BCBS, 2003) since then F_D^* is simplified to

$$F_D^*(z) = LGD \qquad \text{for all } z \in (0, 1). \tag{2.7b}$$

In Section 3, we will show with an example how F_D^* can be chosen in order to incorporate correlation effects into the Basel risk weight functions.

With recourse to Gauss quadrature the integral in (2.6b) can be efficiently approximated with only five addends (cf. Martin et al., 2001). Observe that with the change of variable

$$t = -1 + 2\, p^{-1} \big(\Phi(\sqrt{\rho}\, x + \sqrt{1-\rho}\, z) - 1 + p\big) \tag{2.8}$$

the integral on the right-hand side of (2.6b) can be written as

$$\mathrm{E}[L \,|\, X = x] = \frac{p}{2\sqrt{1-\rho}} \int_{-1}^{1} H(t, x)\, dt \tag{2.9a}$$

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with

$$H(t,x) = \frac{\varphi\left(\frac{\Phi^{-1}(p\,(t+1)/2+1-p)-\sqrt{\rho}\,x}{\sqrt{1-\rho}}\right)}{\varphi\left(\Phi^{-1}(p\,(t+1)/2)+1-p\right)}\, F_D^*\big((t+1)/2\big). \qquad (2.9b)$$

Representation (2.9a), (2.9b) is appropriate for a Gauss-approximation with weights and sampling points derived from the Legendre polynomials (cf. Stoer, 1983). In particular, an approximation of $\mathrm{E}[L\,|\,X=x]$ with five weights and sampling points is given by

$$\mathrm{E}[L\,|\,X=x] \;\approx\; \frac{p}{2\sqrt{1-\rho}}\sum_{i=1}^{5} w_i\,H(t_i,x), \qquad (2.10a)$$

where the numbers w_i and t_i are specified as

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\begin{aligned}
t_1 &= -0.9061798459, & w_1 &= 0.2369268851, \\
t_2 &= -0.5384693101, & w_2 &= 0.4786286705, \\
t_3 &= 0, & w_3 &= 128/225, \qquad\qquad (2.10b) \\
t_4 &= -t_2, & w_4 &= w_2, \\
t_5 &= -t_1, & w_5 &= w_1.
\end{aligned}
$$

3 Calibrating the model

According to the current Basel II philosophy[2], the capital charge[3] as a percentage of the total exposure for the percentage loss variable L will be calculated as

$$\mathrm{Charge}(L) \;=\; \mathrm{E}[L\,|\,X=\Phi^{-1}(\alpha)] - \mathrm{E}[L], \qquad (3.1)$$

where α denotes some high confidence level (currently $\alpha = 0.999$). In order to evaluate (3.1), according to (2.6a) and (2.6b), we have to specify values for the probability of default p and the asset correlation ρ as well as to provide an appropriate functional form for the distribution function F_D of the realized losses.

The parameters p and ρ can be estimated (or in case of ρ set by the supervisors) as usual in the Basel II framework. In principle, F_D can be estimated parametrically or non-parametrically from a sample of realized losses. We suggest to choose a Beta-distribution[4] as a parametric representation of F_D and to determine its shape parameters a and b via moment matching from

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[2]Charging capital only for the *unexpected loss* as expressed by the difference of a high confidence level quantile of the loss and the expected loss (BCBS, 2004).

[3]For the sake of clarity, in this note we do not mention *exposures at default* and *maturity adjustments*. However, the expressions for the capital charges only have to be multiplied with the corresponding factors from the Basel II model in order to cover the full scope of an extended Basel II model.

[4]The same calculations as those presented in Section 4 were carried out with normal and gamma distributions instead of the Beta-distribution. The observed differences in the resulting capital charges were negligible (less than 2.5% of the Basel II charges).

estimates LGD of the expectation of F_D and $VLGD$ of the variance of F_D. The general density of a Beta-distribution is given by

$$\beta(a, b; x) \;=\; \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\, x^{a-1}\,(1-x)^{b-1}, \; 0 < x < 1, \tag{3.2}$$

where Γ denotes the Gamma-function expanding the factorial function to the positive real numbers. For given values of LGD and $VLGD$ the parameters a and b then can be calculated by

$$a = \frac{LGD}{VLGD}\left(LGD\,(1-LGD) - VLGD\right) \tag{3.3a}$$

and

$$b = \frac{1 - LGD}{VLGD}\left(LGD\,(1-LGD) - VLGD\right). \tag{3.3b}$$

For regulatory purposes, it might be adequate to prescribe conservative values for $VLGD$ instead of allowing to insert statistical estimates into (3.3a) and (3.3b). This can be conveniently afforded by specifying $VLGD$ as a fixed percentage v of the maximally possible variance $LGD\,(1-LGD)$ of F_D. This leads to the representations

$$a = LGD\,\frac{1 - v}{v} \tag{3.4a}$$

and

$$b = (1 - LGD)\,\frac{1 - v}{v}. \tag{3.4b}$$

In current credit portfolio models, $v = 0.25$ seems to be a common choice that we will adopt for the numerical examples in Section 4. However, other values of v might turn out to be more appropriate. Moreover, as with the asset correlation, v could be modelled as a function of the probability of default or of the expected loss given default.

4 Numerical examples

In order to illustrate the effect caused by replacing the current Basel II formulas for the capital charges with (2.6a) and (2.6b), we calculated capital charges with both approaches as well as with the approximation formula (2.10a) as functions of the probability of default and of the expected loss given default respectively. As described in Section 3 we chose a Beta-distribution for modelling the distribution of the loss given default rates. The parameters of this Beta-distribution were fixed by (3.4a) and (3.4b) with $v = 0.25$. The correlations involved in (2.6b) were set according to the rule for corporate exposures (BCBS, 2003), i.e.

$$\rho \;=\; \rho(p) \;=\; \frac{1 - e^{-50\,p}}{1 - e^{-50}}\, 0.12 + \left(1 - \frac{1 - e^{-50\,p}}{1 - e^{-50}}\right) 0.24. \tag{4.1}$$

The calculations in case of the single risk factor approach were exact in the sense of being carried out with a high-precision numerical integration routine.

PD	Basel II charge	Single risk factor charge	Approx. single risk factor charge
0.03	0.6	0.7	0.7
0.1	1.4	1.7	1.7
0.25	2.8	3.2	3.2
0.5	4.2	4.9	4.8
0.75	5.1	6.1	6.0
1.0	5.9	7.0	6.8
2.0	7.7	9.3	9.1
3.0	8.8	10.8	10.5
5.0	10.6	13.2	12.8
7.5	12.5	16.0	15.4
10.0	14.1	18.4	17.7
15.0	16.4	22.5	21.5
20.0	17.8	25.5	24.2
100.0	0.0	25.9	20.6

Table 1: Capital charges according to Basel II and single risk factor approach as functions of PD. Expected LGD fixed at 45%. LGD variance fixed at 25% of maximally possible variance. Maturity fixed at one year. Asset correlations calculated according to the Basel II corporate curve. All figures in %.

Table 1 lists the results in the case of varying PDs with fixed expected LGD. Table 2 gives the results in the case of fixed PD with varying expected LGD. In particular, the tabulated numbers show that the relative difference of the Basel charges and the "single risk factor" charges increases in PD and decreases in expected LGD. Differences diminish to zero when the expected LGD approaches 100%. With the exception of the 100% PD case, the quality of the (2.10a) approximation is satisfactory. Even in this case, the resulting charge of 20% is much more realistic than the 0% Basel II charge.

5   Conclusions

We have introduced a new methodology for incorporating LGD correlation effects into the Basel II capital charge formulas. The new methodology is based on a single risk factor modelling of LGD and default event. Compared to the current capital charge formulas, the new formulas require the specification of one additional parameter, namely the LGD volatility. This LGD volatility can be estimated with standard methods or be prescribed by the supervisors in order to ensure a conservative regulatory capital allocation. By numerical examples, we have illustrated that the difference of the capital charges according to Basel II and according to the single risk factor approach increases with growing PDs and decreases with growing expected LGDs. The integral formulas for the capital charge that are involved with the single risk factor model can be efficiently approximated by a weighted sum – as specified in Equation (2.10a) – with five addends.

LGD	Basel II charge	Single risk factor charge	Approx. single risk factor charge
5	0.7	1.2	1.0
10	1.3	2.1	1.9
15	2.0	2.9	2.7
20	2.6	3.6	3.5
25	3.3	4.4	4.2
30	3.9	5.1	4.9
35	4.6	5.7	5.5
40	5.2	6.4	6.2
45	5.9	7.0	6.8
50	6.5	7.6	7.5
55	7.2	8.3	8.1
60	7.8	8.8	8.7
65	8.5	9.4	9.3
70	9.1	10.0	9.9
75	9.8	10.6	10.4
80	10.4	11.1	11.0
85	11.1	11.6	11.5
90	11.7	12.1	12.0
95	12.4	12.6	12.5
100	13.0	13.0	13.0

Table 2: Capital charges according to Basel II and single risk factor approach as functions of expected LGD. PD fixed at 1.0%. LGD variance calculated as 25% of maximally possible variance. Maturity fixed at one year. Asset correlations calculated according to the Basel II corporate curve. All figures in %.

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A The potential loss model and its connection with the single risk factor model

We present an interpretation of the potential loss model by Pykhtin (2003) that makes the model comparable to the single risk factor model as described by (2.5). For the potential loss model, one assumes that the loss experienced with a fixed obligor is driven by two risk factor random variables: U for toggling the default event and V for settling the amount of loss. Consequently, the loss variable L is defined by

$$L = \begin{cases} G(V), & \text{if } U > c, \\ 0, & \text{otherwise,} \end{cases} \qquad (A.1)$$

where $G(v)$ is function that is non-decreasing in v and c is an appropriate default threshold that ensures $P[U > c] = p$, if p denotes the probability of default. U and V may be interpreted as negative changes in the obligor's assets value at different moments in time, for instance at default and some months after default respectively.

By (2.5), with $U = V = \sqrt{\rho}\,X + \sqrt{1-\rho}\,\xi$ the single risk factor model turns out to be a special case of the potential loss model. Note that $F_D^*\big(\frac{\Phi(\sqrt{\rho}\,X+\sqrt{1-\rho}\,\xi)-1+p}{p}\big) = 0$ on the non-default event $\sqrt{\rho}\,X + \sqrt{1-\rho}\,\xi \le \Phi^{-1}(1-p)$. Hence, in the single risk factor model the amount of loss equals zero if the the obligor survives. This is in contrast to the potential loss model, where V and U may be dependent but need not be identical. As a consequence, $G(V) > 0$ may happen even if $U \le c$, i.e. in case of the obligor's survival. Because of this observation, Pykhtin (2003) called $G(V)$ the *potential loss*.

Pykhtin (2003) (cf. also Frye, 2000) specifies the vector (U, V) as bivariate normal vector via

$$\begin{aligned} U &= \sqrt{\rho}\,X + \sqrt{1-\rho}\,\xi \qquad \text{and} \\ V &= \sqrt{\omega}\,X + \sqrt{1-\omega}\,\eta, \end{aligned} \qquad (A.2)$$

where X, ξ, and η are independent and standard normally distributed and $\rho, \omega \in (0,1)$ are *asset correlations* (cf. Section 2). The constant c in (A.1) is then given by $c = \Phi^{-1}(1-p)$.

By (A.2) and (2.5), the main difference between the single risk factor and the potential loss model can be explained as follows. (2.5) shows that in the single loss model the amount of loss and the default indicator function are *co-monotonous*, i.e. they are connected through the strongest form of stochastic dependence. In contrast, by (A.2) the dependence of the loss amount

$G(V)$ and the default indicator is much weaker. In particular, high values of $G(V)$ may occur even if the indicator takes the value 0, i.e. there is no default.

Pykhtin (2003) suggests to choose G as

$$G(v) = \max\big(1 - \exp(-\mu - \sigma v), 0\big) \tag{A.3}$$

with $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$ fixed. Together with (A.1) and (A.2), (A.3) completely specifies the potential loss model. The loss distribution function F_L in this case reads

$$F_L(t) = \begin{cases} 0, & t < 0, \\ 1 - p + \Phi\big(-\frac{\mu + \log(1-t)}{\sigma}\big) - \Phi_2\big(c, -\frac{\mu + \log(1-t)}{\sigma}; \sqrt{\rho\omega}\big), & 0 \le t < 1, \\ 1, & t \ge 1, \end{cases} \tag{A.4}$$

where $\Phi_2(\,\cdot\,, \,\cdot\,; \tau)$ denotes the standard bivariate normal distribution function with correlation τ. Observe from (A.4) that $\mathrm{P}[L = 0] > 1 - p$ holds in this model.

At first glance, the potential loss model as specified by (A.1), (A.2), and (A.3) appears to be quite simple. However, the realized losses are not observed in a form according to (A.1) and (A.2) but rather via the loss distribution (A.4). This loss distribution contains parameters that have to be estimated in a way that makes estimation a non-trivial task.

The single risk factor model, in contrast to the potential loss model, starts with the observable loss distribution (2.1). Complexity enters this model only in a second step when the conditional expectation $\mathrm{E}[L \,|\, X = x]$ has to be computed (see (2.6b)). This calculation of $\mathrm{E}[L \,|\, X = x]$, in turn, is easy in case of the potential loss model as given by (A.1) and (A.2) since conditional on X independence of the involved factors holds. With this observation in mind, we obtain

$$\mathrm{E}[L \,|\, X = x] = \mathrm{P}\big[\sqrt{\rho}\, X + \sqrt{1-\rho}\, \xi > \Phi^{-1}(1-p) \,|\, X = x\big]\, \mathrm{E}\big[G(\sqrt{\omega}\, X + \sqrt{1-\omega}\, \eta)) \,|\, X = x\big]$$
$$= \Phi\left(\frac{\sqrt{\rho}\, x + \Phi^{-1}(p)}{\sqrt{1-\rho}}\right) \int_{-\infty}^{\infty} G(\sqrt{\omega}\, x + \sqrt{1-\omega}\, z)\, \varphi(z)\, dz. \tag{A.5}$$

Given the specification of G by (A.3), the integral in (A.5) can be explicitly evaluated as

$$\mathrm{E}\big[G(\sqrt{\omega}\, X + \sqrt{1-\omega}\, \eta)) \,|\, X = x\big] = 1 - \Phi\left(-\frac{\mu + \sigma \sqrt{\omega}\, x}{\sigma \sqrt{1-\omega}}\right)$$
$$- \exp\big(-\mu - \sigma \sqrt{\omega}\, x + \sigma^2 (1-\omega)/2\big) \left(1 - \Phi\left(-\frac{\mu + \sigma \sqrt{\omega}\, x}{\sigma \sqrt{1-\omega}} + \sigma \sqrt{1-\omega}\right)\right). \tag{A.6}$$